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Investigation of Stability and Hydrodynamics of Different Lattice Boltzmann Models

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Stability and hydrodynamic behaviors of different lattice Boltzmann models including the lattice Boltzmann equation (LBE), the differential lattice Boltzmann equation (DLBE), the interpolation-supplemented lattice Boltzmann method (ISLBM) and the Taylor series expansion- and least square-based lattice Boltzmann method (TLLBM) are studied in detail. Our work is based on the von Neumann linearized stability analysis under a uniform flow condition. The local stability and hydrodynamic (dissipation) behaviors are studied by solving the evolution operator of the linearized lattice Boltzmann equations numerically. Our investigation shows that the LBE schemes with interpolations, such as DLBE, ISLBM and TLLBM, improve the numerical stability by increasing hyper-viscosities at large wave numbers (small scales). It was found that these interpolated LBE schemes with the upwind interpolations are more stable than those with central interpolations because of much larger hyper-viscosities.

KEY WORDS: Stability analysis; hydrodynamics; Lattice Boltzmann equation (LBE); ISLBM; TLLBM; DLBE.

1. INTRODUCTION

Since the lattice Boltzmann method (LBM) was introduced in the late 1980s, several kinds of lattice Boltzmann approaches[(1–6)] have been developed, based on the fact that the formal connections exist between the lattice Boltzmann equation (LBE) and the continuous Boltzmann equation,[(7,8)] and the particle distribution function is continuous in physical

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space. However, despite the notable success of these approaches in many applications,[9] systematic investigations on their stability and hydrodynamic behaviors are still demanding.

In the continuum Boltzmann equation, the Maxwell equilibrium distribution function corresponds to the maximum entropy state, and hence the continuum Boltzmann equation is completely stable in terms of the Boltzmann's H-theorem.[9,10] However, in the LBE, since only a small set of discrete velocities is used and the equilibrium distribution function is usually derived from the Maxwell equilibrium distribution function by the truncated Taylor series expansion, the H-theorem is no longer satisfied.[11] As a consequence, the LBE is subjected to numerical instability. On the other hand, the LBE is originated from the lattice gas cellular automata (LGCA),[12] but the equilibrium distribution function and the collision operator in the LBE do not need to follow the original LGCA and have more freedom in the choice of their entities[13,14]. Hence, it is important to investigate the hydrodynamic behaviors of these entities so that the numerical data obtained by the LBE simulations has an unambiguous interpretation.

In the past years, intensive studies on the stability and hydrodynamics analysis about lattice Boltzmann models have been carried out by various researchers.[10,15–18] The most systematic investigation among them should be attributed to the work of Lallmand and Luo[17,18] based on the generalized LBE. Generally, the global stability condition of the LBE requires that the mean flow velocity is below a maximum velocity which is a function of several parameters, including the sound speed, the relaxation time and the wave number $\mathbf{k}(k, \phi)$. Behrend et al.[16] and Lallmand and Luo[17] suggested that the LBE with multiple relaxations is more stable than that with single relaxation.

The analysis of the hydrodynamic behaviors of the LBE is a little bit more complicated. The LBE with a single relaxation has a good hydrodynamic behavior with the relaxation parameter $\tau \leqslant 1$ [16], and a better hydrodynamic behavior can be achieved for the LBE with multiple relaxations.[16,17] Lallmand and Luo[17] also compared the hydrodynamic behaviors of the LBE and the interpolation-supplemented LBM (ISLBM),[4] and showed that the ISLBM creates a large amount of numerical viscosity when comparing with the LBE.

In this paper, we focus on the comparative study of the local numerical stability analysis and hydrodynamic behaviors of different lattice Boltzmann approaches with a single relaxation published so far, including the LBE [1], the differential LBE (DLBE),[6] the ISLBM[4] and the Taylor-series-expansion- and least-squares-based LBM (TLLBM).[5] In the analyses of hydrodynamics, the main focus is on the dissipation, which is an important index of the hydrodynamic behaviors and demonstrated as **k**-dependent viscosity. In this work, all analyses are based on the linearized eigenvalue equation system.

In Section 2, we will give a brief introduction to different lattice Boltzmann approaches based on the two-dimensional (2D) nine-bit velocity model (D2Q9). Section 3 shows linearized equations for the different lattice Boltzmann approaches. Comparisons and discussion of the stability and hydrodynamic behaviors of the different lattice Boltzmann approaches are presented in Section 4. Section 5 provides a summary and then concludes the paper.

2. LATTICE BOLTZMANN APPROACHES BASED ON THE D2Q9 MODEL

The principle of the LBM is to construct a dynamic system on a lattice involving a number of the single-particle distribution functions of fictitious particles. The particles then evolve in a discrete time according to certain rules that guarantee the satisfaction of some desirable macroscopic behaviors. To simplify our analysis, we here confine our study to the 2D isothermal flow with uniform velocity field.

2.1. D2Q9 Model and LBE

The D2Q9 discrete velocity model defined in a square lattice is

$$e_\alpha = \begin{cases} (0,0), & \alpha=0, \\ (\cos[(\alpha-1)\pi/2], \sin[(\alpha-1)\pi/2])c, & \alpha=1,2,3,4, \\ (\cos[(\alpha-5)\pi/2+\pi/4], \sin[(\alpha-5)\pi/2+\pi/4])\sqrt{2}c, & \alpha=5,6,7,8, \end{cases} \tag{1}$$

where $c=\delta x/\delta t$, and δx and δt are the lattice constant and the time step, respectively.

The 2D LBE corresponding to the above model is

$$f_\alpha(x+e_{\alpha x}\delta t, y+e_{\alpha y}\delta t, t+\delta t) = f_\alpha(x,y,t) + \frac{f_\alpha^{\text{eq}}(x,y,t)-f_\alpha(x,y,t)}{\tau}, \quad \alpha=0,1,\ldots,8, \tag{2}$$

where τ is the single relaxation parameter; f_α is the density distribution function along the α direction; δt is the time step and $\mathbf{e}_\alpha(e_{\alpha x}, e_{\alpha y})$ is the particle velocity in the α direction. f_α^{eq} is its corresponding equilibrium state, which depends on the local macroscopic variables such as density ρ and velocity $\mathbf{U}(u, v)$ and is given for isothermal fluids as

$$f_\alpha^{eq} = w_\alpha \rho \left[1 + 3\frac{(\mathbf{e}_\alpha \cdot \mathbf{U})}{c^2} + \frac{9}{2}\frac{(\mathbf{e}_\alpha \cdot \mathbf{U})^2}{c^4} - \frac{3}{2}\frac{\mathbf{U}^2}{c^2} \right] \tag{3}$$

with $w_0 = 4/9$, $w_\alpha = 1/9$ for $\alpha = 1, \ldots, 4$, and $w_\alpha = 1/36$ for $\alpha = 5, \ldots, 8$. The macroscopic density ρ and the velocity $\mathbf{U}$ are obtained from

$$\rho = \sum_{\alpha=0}^{8} f_\alpha, \qquad \rho\mathbf{U} = \sum_{\alpha=0}^{8} f_\alpha \mathbf{e}_\alpha. \tag{4}$$

One of the shortcomings of the standard LBE is that the LBE must be strictly confined in the uniform lattice,(19,20) which makes it like a Cartesian solver in simulation. Since most of the flow problems involve in complex geometries, an irregular grid is always desirable because the boundaries can be described accurately and computational resources can be used efficiently. In the following, we will introduce some other lattice Boltzmann models, which have been demonstrated to overcome the above lattice-uniformity limitation of the standard LBE.

2.2. Differential Lattice Boltzmann Equation

Based on the fact that the distribution function f_α is continuous in the physical space, the differential LBE (DLBE) is directly derived from the standard LBE by using the second order Taylor series expansion in space.(6) The 2D DLBE has the following form:

$$\begin{aligned}
& f_\alpha(x+\delta x, y+\delta y, t+\delta t) \\
& + Cx\frac{\partial f_\alpha(x+\delta x, y+\delta y, t+\delta t)}{\partial x} + Cy\frac{\partial f_\alpha(x+\delta x, y+\delta y, t+\delta t)}{\partial y} \\
& + \frac{1}{2}(Cx)^2\frac{\partial^2 f_\alpha(x+\delta x, y+\delta y, t+\delta t)}{\partial x^2} + \frac{1}{2}(Cy)^2\frac{\partial^2 f_\alpha(x+\delta x, y+\delta y, t+\delta t)}{\partial y^2} \\
& + CxCy\frac{\partial^2 f_\alpha(x+\delta x, y+\delta y, t+\delta t)}{\partial x \partial y} \\
& = f_\alpha(x, y, t) + \left[f_\alpha^{eq}(x, y, t) - f_\alpha(x, y, t) \right]/\tau,
\end{aligned} \tag{5}$$

where $Cx = e_{\alpha x}\delta t - \delta x$ and $Cy = e_{\alpha y}\delta t - \delta y$.

Apparently, the DLBE has no limitation on the grid type, and can be solved by FD or FV method. In the following stability and hydrodynamic analyses, the FD method is used for simplicity.

2.3. Interpolation-Supplemented LBM

The ISLBM was first proposed by He et al. in 1996.[(4)] The basic idea of the ISLBM is to interpolate the distribution function f_α from a fine mesh to a coarse mesh in order to improve the spatial resolution and overcome the limitation of the lattice uniformity. Comparing to the LBE, the ISLBM alters the advection step by a second-order interpolation scheme while keeping the collision step unchanged. In the general curvilinear coordinate system (ξ, η), the 2D governing equation in the ISLBM can be given as

$$f_{\alpha:i,j}(t+\delta t)=\sum_{m=0}^{2}\sum_{l=0}^{2}c_{i,m}c_{j,l}g_{\alpha:i+m\times i_d,j+l\times j_d}, \tag{6}$$

where $i_d=\mathrm{sign}(d\xi_i)$ and $j_d=\mathrm{sign}(d\eta_j)$, and $d\xi_i=\xi(x+e_{\alpha x}\delta t, y+e_{\alpha y}\delta t)-\xi_i$, $d\eta_j=\eta(x+e_{\alpha x}\delta t, y+e_{\alpha y}\delta t)-\eta_j$. The $g_{\alpha:i,j}$ is the post collision distribution function that is expressed as $g_{\alpha:i,j}=f_{\alpha:i,j}(t)+(f^{\mathrm{eq}}_{\alpha:i,j}(t)-f_{\alpha:i,j}(t))/\tau$ and $c_{i,m}$ and $c_{j,l}$ are the interpolation coefficients depending on the mesh coordinates which are calculated by

$$\begin{aligned}
&c_{i,1}=(|d\xi_i|-1)(|d\xi_i|-2)/2, \qquad c_{j,1}=(|d\eta_j|-1)(|d\eta_j|-2)/2,\\
&c_{i,2}=-|d\xi_i|(d\xi_i|-2), \qquad c_{j,2}=-|d\eta_j|(|d\eta_j|-2),\\
&c_{i,3}=|d\xi_i|(d\xi_i|-1)/2, \qquad c_{j,3}=|d\eta_j|(d\eta_j|-1)/2
\end{aligned} \tag{7}$$

for the upwind interpolation scheme (US), and

$$\begin{aligned}
&c_{i,1}=1-|d\xi_i|^2, \quad c_{j,1}=1-\left|d\eta_j\right|^2,\\
&c_{i,2}=|d\xi_i|(|d\xi_i|+1)/2, \quad c_{j,2}=|d\eta_j|(|d\eta_j|+1)/2,\\
&c_{i,3}=|d\xi_i|(|d\xi_i|-1)/2, \quad c_{j,3}=|d\eta_j|(|d\eta_j|-1)/2
\end{aligned} \tag{8}$$

for the central interpolation scheme (CS).

2.4. The Taylor-Series-Expansion and Least-Squares-Based LBM (TLLBM)

The TLLBM developed by Shu et al. [5] is based on the standard LBE, the well-known Taylor series expansion, the idea of developing Runge–Kutta method,(21) and the least-squares approach.(22) The final form of the 2D TLLBM is

$$f_\alpha(x_0, y_0, t+\delta t) = \sum_{k=1}^{M+1} a_{1,k} g_{\alpha:k-1}, \tag{9}$$

where M is the number of neighboring points in which the Taylor series expansion is applied, $a_{1,k}$ are the elements of the first row of the matrix $[A]$, which is given as

$$[A] = ([S]^{\mathrm{T}}[S])^{-1}[S]^{\mathrm{T}}. \tag{10}$$

Here, $[S]$ is a $(M+1)\times 6$ dimensional matrix, which is given as

$$[S] = \begin{bmatrix} 1 & \Delta x_0 & \Delta y_0 & (\Delta x_0)^2/2 & (\Delta y_0)^2/2 & \Delta x_0 \Delta y_0 \\ 1 & \Delta x_1 & \Delta y_1 & (\Delta x_1)^2/2 & (\Delta y_1)^2/2 & \Delta x_1 \Delta y_1 \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ 1 & \Delta x_M & \Delta y_M & (\Delta x_M)^2/2 & (\Delta y_M)^2/2 & \Delta x_M \Delta y_M \end{bmatrix}_{(M+1)\times 6} \tag{11}$$

where $\Delta x_i = x_i + e_{\alpha x}\delta t - x_0$, $\Delta y_i = y_i + e_{\alpha y}\delta t - y_0$ and $g_{\alpha:l}$ has the same definition as that in the ISLBM.

3. LINEARIZATION OF DIFFERENT LATTICE BOLTZMANN MODELS

To analyze the stability and hydrodynamic behaviors of the different lattice Boltzmann models, the linearization of their formulations is first conducted. It is noted that for all versions of LBM, the non-linearity is appeared in the collision term. In other words, it is appeared in the equilibrium function f_α^{eq} or the post collision function g_α. When f_α is changed by f'_α, g_α will be changed by

$$g'_\alpha = \sum_\beta \frac{\partial g_\alpha}{\partial f_\beta} f'_\beta = \left(1 - \frac{1}{\tau}\right) f'_\alpha + \frac{1}{\tau} \sum_\beta J_{\alpha\beta} f'_\beta, \tag{12}$$

where $J_{\alpha\beta} = \partial f_\alpha^{eq}/\partial f_\beta$ and $\partial f_\alpha^{eq}/\partial f_\beta$ is defined as

$$\frac{\partial f_\alpha^{eq}}{\partial f_\beta} = w_\alpha \Bigg(1 + \frac{\mathbf{e}_\beta \cdot \mathbf{e}_\alpha}{c_s^2} + \frac{1}{2c_s^2} \Big\{ \Big[2(\mathbf{e}_\beta \cdot \mathbf{e}_\alpha) \cdot (\mathbf{e}_\alpha \cdot \mathbf{U}) - (\mathbf{e}_\alpha \cdot \mathbf{U})^2 \Big] + \big[2(\mathbf{e}_\beta \cdot \mathbf{U}) - \mathbf{U} \cdot \mathbf{U} \big] \Big\} \Bigg). \tag{13}$$

The coefficient w_α is the same as that used in Eq (3). With Eq. (12), all the lattice Boltzmann formulations become linear about f'_α. In the following, we will perform the Von Neumann stability analysis and determine the advection operator of each lattice Boltzmann model.

3.1. Von Neumann Analysis of the LBE

With Eq (12), the linearization of the LBE can be written as

$$f'_\alpha(\mathbf{r} + \mathbf{e}_\alpha \delta_t, t + \delta_t) = (1 - \frac{1}{\tau}) f'_\alpha(\mathbf{r}, t) + \frac{1}{\tau} \sum_\alpha J_{\beta\alpha}(\mathbf{r}) f'_\alpha(\mathbf{r}, t). \tag{14}$$

Equation (14) can be put into the following matrix form:

$$\mathbf{f}'(\mathbf{r} + \mathbf{e}_\alpha \delta_t, t + \delta_t) = [(1 - \frac{1}{\tau})\mathbf{I} + \frac{1}{\tau}\mathbf{J}]\mathbf{f}'(\mathbf{r}, t), \tag{15}$$

where $\mathbf{I}$ is the unit diagonal matrix, $\mathbf{f}'(\mathbf{r}, t) = [f'_0(\mathbf{r}, t), f'_1(\mathbf{r}, t), \hbar, f'_8(\mathbf{r}, t)]^T$, $\mathbf{J}_{\alpha\beta} = J_{\alpha\beta}(\mathbf{r})$.

In the Fourier space $[-\pi, \pi]$, the state of the system is represented as $f'_\alpha(\mathbf{r}, t) = \sum_{\mathrm{k}} f'_\alpha(\mathbf{k}) e^{\mathrm{ik \cdot r}}$ and the above equation becomes:

$$\mathbf{f}'(\mathbf{k}, t + \delta_t) = \mathbf{L}(\mathbf{k}) \mathbf{f}'(\mathbf{k}, t), \tag{16}$$

where $\mathbf{L}(\mathbf{k}) = \mathbf{A}^{-1}[(1 - 1/\tau)\mathbf{I} + 1/\tau \mathbf{J}]$ is the evolution operator and $\mathbf{A}$ is the advection operator denoted by

$$\mathbf{A}_{\alpha\beta} = e^{i(\mathrm{k \cdot e_\alpha} \delta_t)} \delta_{\alpha\beta}. \tag{17}$$

Here $\delta_{\alpha\beta}$ is the Kronecker delta. For other LBE approaches introduced in Section 2, the analysis is similar to the LBE. Since the collision step of these LBE approaches remains unchanged, one only needs to correct their advection operators.

3.2. Advection Operator of the DLBE

There are several possibilities[23] to approximate the spatial derivatives in the ordinary differential equation like the DLBE. In the following, only the second-order central and upwind difference schemes are used for numerical discretization. In the general curvilinear coordinate system, Eq (5) can be discretized as

$$\begin{aligned}
&f_{\alpha:i,j}^{n+1}+\frac{mi}{2}A(3f_{\alpha:i,j}^{n+1}-4f_{\alpha:i-mi,j}^{n+1}+f_{\alpha:i-2mi,j}^{n+1})\\
&+\frac{mj}{2}B(3f_{\alpha:i,j}^{n+1}-4f_{\alpha:i,j-mj}^{n+1}+f_{\alpha:i,j-2mj}^{n+1})\\
&+\frac{C}{2}(f_{\alpha:i+1,j}^{n+1}-2f_{\alpha:i,j}^{n+1}+f_{\alpha:i-1,j}^{n+1})+\\
&\frac{D}{2}(f_{\alpha:i,j+1}^{n+1}-2f_{\alpha:i,j}^{n+1}+f_{\alpha:i,j-1}^{n+1})\\
&+\frac{1}{4}E(f_{\alpha:i+1,j+1}^{n+1}-f_{\alpha:i+1,j-1}^{n+1}-f_{\alpha:i-1,j+1}^{n+1}+f_{\alpha:i-1,j-1}^{n+1})\\
&=f_{\alpha:i-ni,j-nj}^{n}+\left[f_{\alpha:i-ni,j-nj}^{eq,n}-f_{\alpha:i-ni,j-nj}^{n}\right]/\tau
\end{aligned} \tag{18a}$$

by the upwind scheme and

$$\begin{aligned}
&f_{\alpha:i,j}^{n+1}+\frac{1}{2}A(f_{\alpha:i+1,j}^{n+1}-f_{\alpha:i-1,j}^{n+1})+\frac{1}{2}B(f_{\alpha:i,j+1}^{n+1}-f_{\alpha:i,j-1}^{n+1})\\
&+\frac{C}{2}(f_{\alpha:i+1,j}^{n+1}-2f_{\alpha:i,j}^{n+1}+f_{\alpha:i-1,j}^{n+1})+\\
&\frac{D}{2}(f_{\alpha:i,j+1}^{n+1}-2f_{\alpha:i,j}^{n+1}+f_{\alpha:i,j-1}^{n+1})\\
&+\frac{1}{4}E(f_{\alpha:i+1,j+1}^{n+1}-f_{\alpha:i+1,j-1}^{n+1}-f_{\alpha:i-1,j+1}^{n+1}+f_{\alpha:i-1,j-1}^{n+1})\\
&=f_{\alpha:i-ni,j-nj}^{n}+\left[f_{\alpha:i-ni,j-nj}^{eq,n}-f_{\alpha:i-ni,j-nj}^{n}\right]/\tau
\end{aligned} \tag{18b}$$

by the central difference scheme, respectively. Here the coefficients A, B, C, D and E are dependent of Cx and Cy, which are defined as:[5]

$$\begin{aligned}
&A=\alpha+\frac{1}{2}\alpha^2 J_{\eta\varsigma\xi}+\alpha\beta J_{\eta\varsigma\eta}+\frac{1}{2}\beta^2 J_{\eta\eta\eta},\\
&B=\beta+\frac{1}{2}\alpha^2 J_{\xi\varsigma\xi}+\alpha\beta J_{\xi\varsigma\eta}+\frac{1}{2}\beta^2 J_{\xi\eta\eta},\\
&C=\frac{1}{2}\alpha^2,\\
&D=\frac{1}{2}\beta^2,\ E=\alpha\beta,
\end{aligned}$$

where

$$\alpha = [y_\eta Cx - x_\eta Cy]/J, \qquad \beta = [-y_\xi Cx + x_\xi Cy]/J,$$
$$J_{\eta\xi\xi} = (x_\eta y_{\xi\xi} - x_{\xi\xi} y_\eta)/J, \qquad J_{\eta\eta\eta} = (x_\eta y_{\eta\eta} - x_{\eta\eta} y_\eta)/J,$$
$$J_{\xi\xi\xi} = -(x_\xi y_{\xi\xi} - x_{\xi\xi} y_\xi)/J, \qquad J_{\xi\xi\eta} = -(x_\xi y_{\xi\eta} - x_{\xi\eta} y_\xi)/J,$$
$$J_{\eta\xi\eta} = -(x_\xi y_{\eta\eta} - x_{\eta\eta} y_\xi)/J.$$

The advection operator of the DLBE is

$$\begin{aligned} \mathrm{A}|^{\mathrm{upwind}} = \; & [1 + \frac{mi}{2}\mathrm{A}(3 - 4p_i^{-mi} + p_i^{-2mi}) + \frac{mj}{2}B(3 - 4q_j^{-mj} + q_j^{-2mj}) \\ & + \frac{C}{2}(p_i - 2 + p_i^{-1}) + \frac{D}{2}(q_j - 2 + q_j^{-1}) \\ & + \frac{1}{4}E(p_i q_j - p_i q_j^{-1} - p_i^{-1} q_j + p_i^{-1} q_j^{-1})] \\ & \mathrm{e}^{\mathrm{i}[k_\xi(\xi_{i,j} - \xi_{i-ni,j-nj}) + k_\eta(\eta_{i,j} - \eta_{i-ni,j-nj})]}\delta_{\alpha\beta}, \end{aligned} \tag{19a}$$

$$\begin{aligned} \mathrm{A}|^{\mathrm{centered}} = \; & [1 + \frac{1}{2}\mathrm{A}(p_i - p_i^{-1}) + \frac{1}{2}B(q_j - q_j^{-1}) \\ & + \frac{C}{2}(p_i - 2 + p_i^{-1}) + \frac{D}{2}(q_j - 2 + q_j^{-1}) \\ & + \frac{1}{4}E(p_i q_j - p_i q_j^{-1} - p_i^{-1} q_j + p_i^{-1} q_j^{-1})] \\ & e^{\mathrm{i}[k_\xi(\xi_{i,j} - \xi_{i-ni,j-nj}) + k_\eta(\eta_{i,j} - \eta_{i-ni,j-nj})]}\delta_{\alpha\beta}, \end{aligned} \tag{19b}$$

where

$$mi = \mathrm{sign}(A), \qquad mj = \mathrm{sign}(B),$$
$$p_i = e^{\mathrm{i}k_\xi(\xi_{i+1,j} - \xi_{i,j})}, \qquad q_j = e^{ik_\eta(\eta_{i,j+1} - \eta_{i,j})}, \qquad k_\xi = k_x x_\xi + k_y y_\xi,$$
$$k_\eta = k_x x_\eta + k_y y_\eta, \qquad ni = \mathrm{sign}(e_{\alpha\xi}), nj = \mathrm{sign}(e_{\alpha\eta}),$$
$$e_{\alpha\xi} = [e_{\alpha x} y_\eta - e_{\alpha y} x_\eta]/J, \qquad e_{\alpha\eta} = [-e_{\alpha x} y_\xi + e_{\alpha y} x_\xi]/J,$$

and J is the Jacobian operator.

3.3. Advection Operators of ISLBM and TLLBM

The TLLBM has the same evolution formulation as the ISLBM, and their central and upwind schemes can be written as

$$\begin{aligned} f_{\alpha:i,j}|^{\mathrm{centered}} = \; & a_{\alpha:1} g_{\alpha:i,j} + a_{\alpha:2} g_{\alpha:i-1,j} + a_{\alpha:3} g_{\alpha:i+1,j} \\ & + a_{\alpha:4} g_{\alpha:i,j-1} + a_{\alpha:5} g_{\alpha:i-1,j-1} + a_{\alpha:6} g_{\alpha:i+1,j-1} \\ & + a_{\alpha:7} g_{\alpha:i,j+1} + a_{\alpha:8} g_{\alpha:i-1,j+1} + a_{\alpha:9} g_{\alpha:i+1,j+1}, \end{aligned} \tag{20a}$$

and

$$f_{\alpha:i,j}|^{\text{upwind}} = a_{\alpha:1}g_{\alpha:i,j} + a_{\alpha:2}g_{\alpha:i-ni,j} + a_{\alpha:3}g_{\alpha:i-2ni,j} + a_{\alpha:4}g_{\alpha:i,j-nj} + a_{\alpha:5}g_{\alpha:i-ni,j-nj} + a_{\alpha:6}g_{\alpha:i-2ni,j-nj} + a_{\alpha:7}g_{\alpha:i,j-2nj} + a_{\alpha:8}g_{\alpha:i-ni,j-nj} + a_{\alpha:9}g_{\alpha:i-2ni,j-2nj}, \tag{20b}$$

where $g_{\alpha:i,j}$ is the post collision distribution function and the $a_{\alpha:l}$ are coefficients depending on the coordinates of the points in space, which can be obtained from Eq. (10) for the TLLBM and Eq. (7) and (8) for the ISLBM.

The corresponding advection operators of the TLLBM and the ISLBM can be written as

$$\begin{aligned} A^{-1}_{\alpha\beta}|^{\text{centered}} = \Big\{ & a_{\alpha:1} + a_{\alpha:2}e^{\mathrm{i}k_x(x_{i-1}-x_i)} + a_{\alpha:3}e^{\mathrm{i}k_x(x_{i+1}-x_i)} \\ & + a_{\alpha:4}e^{\mathrm{i}k_y(y_{j-1}-y_j)} + a_{\alpha:5}e^{\mathrm{i}[k_x(x_{i-1}-x_i)+k_y(y_{j-1}-y_j)]} \\ & + a_{\alpha:6}e^{\mathrm{i}[k_x(x_{i+1}-x_i)+k_y(y_{j-1}-y_j)]} \\ & + a_{\alpha:7}e^{\mathrm{i}k_y(y_{j+1}-y_j)} + a_{\alpha:8}e^{\mathrm{i}[k_x(x_{i-1}-x_i)+k_y(y_{j+1}-y_j)]} \\ & + a_{\alpha:9}e^{\mathrm{i}[k_x(x_{i+1}-x_i)+k_y(y_{j+1}-y_j)]}\Big\}\delta_{\alpha\beta}, \end{aligned} \tag{21a}$$

$$\begin{aligned} A^{-1}_{\alpha\beta}|^{\text{upwind}} = \Big\{ & a_{\alpha:1} + a_{\alpha:2}e^{\mathrm{i}k_x(x_{i-ni}-x_i)} + a_{\alpha:3}e^{\mathrm{i}k_x(x_{i-2ni}-x_i)} \\ & + a_{\alpha:4}e^{ik_y(y_{j-nj}-y_j)} + a_{\alpha:5}e^{\mathrm{i}[k_x(x_{i-ni}-x_i)+k_y(y_{j-nj}-y_j)]} \\ & + a_{\alpha:6}e^{\mathrm{i}[k_x(x_{i-2ni}-x_i)+k_y(y_{j-nj}-y_j)]} \\ & + a_{\alpha:7}e^{\mathrm{i}k_y(y_{j-2nj}-y_j)} + a_{\alpha:8}e^{i[k_x(x_{i-ni}-x_i)+k_y(y_{j-nj}-y_j)]} \\ & + a_{\alpha:9}e^{\mathrm{i}[k_x(x_{i-2ni}-x_i)+k_y(y_{j-2nj}-y_j)]}\Big\}\delta_{\alpha\beta}. \end{aligned} \tag{21b}$$

4. LOCAL STABILITY AND HYDRODYNAMIC BEHAVIORS OF DIFFERENT LBE APPROACHES

Numerical stability and hydrodynamic analysis can be made by analyzing the eigenvalues of the linearized evolution operator $\mathbf{L}(\mathbf{k})$ in Eq. (16). However, Eq. (16) cannot be solved analytically in general, except for some special cases. Nevertheless, it can be easily solved using the knowledge of linear algebra. When $\mathbf{k} = 0$, the eigenvalues of the 9×9 matrix $\mathbf{L}(0)$ can be obtained analytically. The matrix of $\mathbf{L}(0)$ has three unit eigenvalues corresponding to the three hydrodynamic (conserved) modes: one transverse and two longitudinal modes. $\mathbf{L}(0)$ also has six non-unit eigenvalues corresponding to six kinetic modes. For finite $\mathbf{k}$, the stability and hydrodynamic behaviors depend on $\mathbf{k}$. A wave-vector-dependent kinematic shear viscosity[24] is defined as

$$\upsilon(\mathbf{k}) = -\frac{\mathrm{Re}[\ln z_{\perp}(\mathbf{k})]}{k^2}. \tag{22}$$

From Eq. (22), the instability can be easily identified by

$$\mathrm{Re}[\ln z_{\alpha}(\mathbf{k})] > 0, \tag{23}$$

where $z_{\alpha}(\mathbf{k})$ are the eigenvalues corresponding to the different kinetic modes of the matrix $\mathbf{L}$ $(\mathbf{k})$ and $z_{\perp}(\mathbf{k})$ is the eigenvalue corresponding to the shear hydrodynamic mode of the matrix $\mathbf{L}$ $(\mathbf{k})$.

For the local analysis in the following part, we arbitrarily choose a 51×51 grid with uniform and non-uniform distributions in a square domain. The analysis is the same at any point for the uniform grid. For the non-uniform grid, the analysis is carried out at two positions with the grid stretch ratios of $r = \delta r / \delta r_{\min} = 1.09$ and 1.25, respectively. Since there can be many kinds of non-uniform grids, in this work, we only choose one arbitrarily and compare the stability and hydrodynamic behaviors of the different LBE approaches.

4.1. Hydrodynamic Behaviors of Different LBE Approaches

The hydrodynamic behaviors of the different LBE approaches were studied in the whole k space and their representative plots of the normalized shear viscosity $\upsilon(\mathbf{k})/\upsilon(0)$ were shown in Figs. 1 and 2, where a uniform flow with zero velocity was assumed and the relaxation parameter $\tau = 0.55$ was used. Fig. 1 shows the results at three phase angles of $\phi = 0, \pi/8, \pi/4$ based on the uniform grid. In this case, the LBE, the DLBE and the ISLBM have the same shear hydrodynamic behaviors because their eveloution equations are exactly the same. It was found in Fig. 1 that, the TLLBM introduces about one order higher of numerical viscosity than those of other LBE approaches at large k, and the TLLBM with the central scheme has a slighlty better shear hydrodynamic behavior than that with the upwind one. When the non-uniform grid was used, the DLBE, the ISLBM and the TLLBM all introduce an enormous numerical viscosity (see Fig. 2, all results are based on the central scheme and at two different space positions with the grid stretch ratios of $r = 1.09$ and 1.25, respectively; $\phi = \pi/8$), but the amount introduced in the DLBE is much smaller than those in the ISLBM and the TLLBM. With the same grid stretch, the numerical viscosity introduced in the ISLBM and the TLLBM are almost the same. On the other hand, the larger the grid stretch, the larger the hyper-viscosities are introduced by interpolations used in the DLBE, ISLBE and TLLBE.

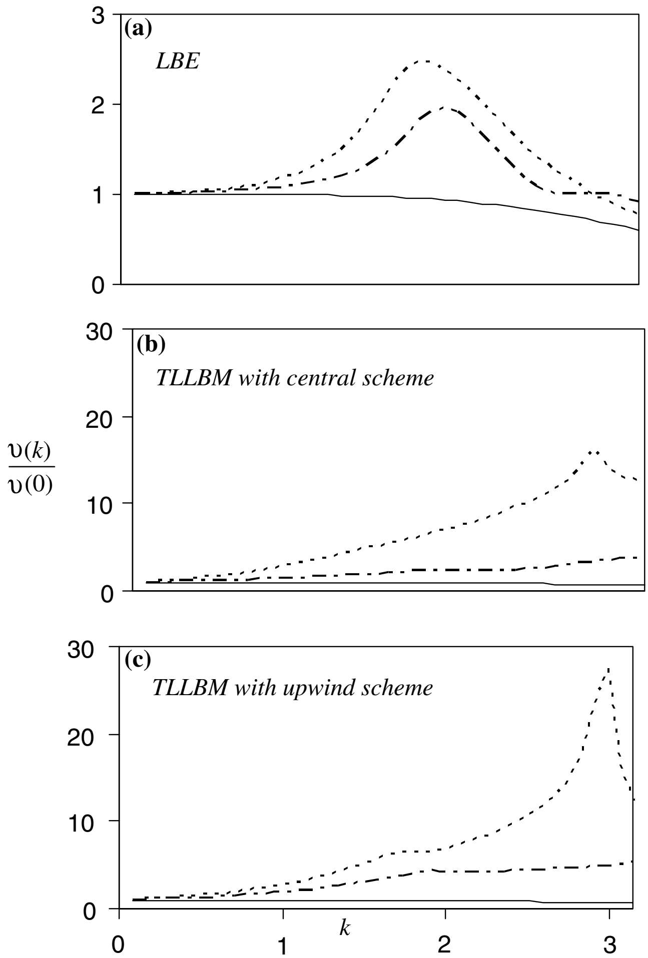

Fig. 1. **k** dependence of viscosity of the different LBE approaches on the uniform grid (——) $\phi=0$; (— - - —) $\phi=\frac{\pi}{8}$; (- - - -) $\phi=\frac{\pi}{4}$. Note that the DLBE and ISLBM reduce to the LBE under uniform grid ($r=1$), the representative plots of the LBE are only shown in (a).

4.2. Stability Behaviors of Different LBE Approaches

The stabilities of the LBE approaches are also related to the hydrodynamic behaviors. Previous stability study[14] has shown that, in the whole **k** space, the stability of the LBE based on the D2Q9 lattice model deteriorate as τ close to 1/2 in the u–τ plane. In the present paper, we further compared local stability behaviors of the different LBE approaches based

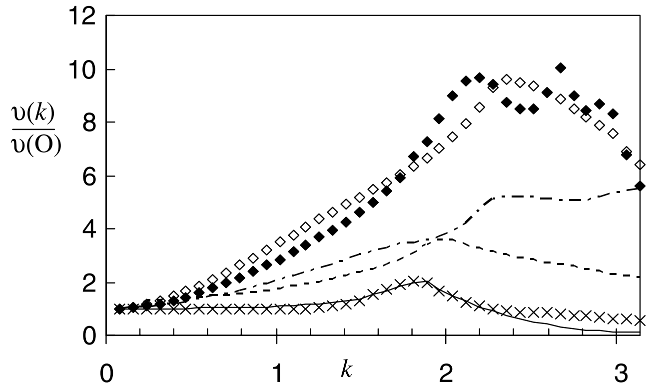

Fig. 2. **k** dependence of viscosity of the different LBE approaches on the non-uniform grid at a phase angle of $\phi = \frac{\pi}{8}$ (——) DLBE ($r = 1.09$); (×) DLBE ($r = 1.25$); (—) ISLBM ($r = 1.09$); ♦ ISLBM ($r = 1.25$); (– - –) TLLBM ($r = 1.09$); ◇ TLLBM ($r = 1.25$).

on the D2Q9 lattice model and different grid distributions. Fig. 3 shows the neutral stability curves of different LBE approaches based on a uniform grid in the u–τ plane. It should be pointed out that, in the uniform grid, the DLBE, the ISLBE and the LBE have the same stable regions because the DLBE and ISLBE are reduced to the LBE, and due to the use of the Runge–Kutta idea and the least-squares theory, the TLLBM has a form different from the LBE. As shown in Fig. 3, it was found that, all the LBE approaches are unstable when $\tau \leqslant 1/2$. Comparing to

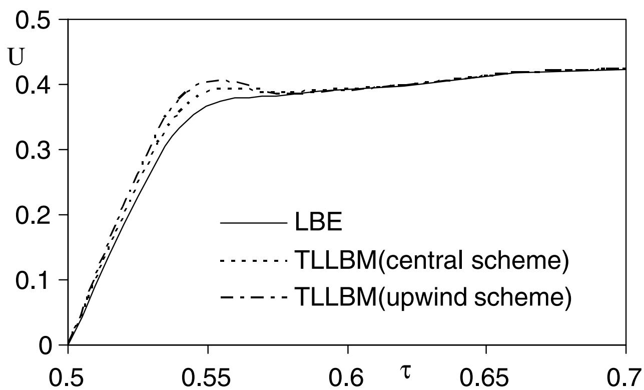

Fig. 3. The neutral stability u–τ curves of different LBE approaches on the uniform grid. (Note that the DLBE and ISLBM reduce to the LBE under uniform grid ($r = 1$), the representative plot of the LBE is only shown in this Figure).

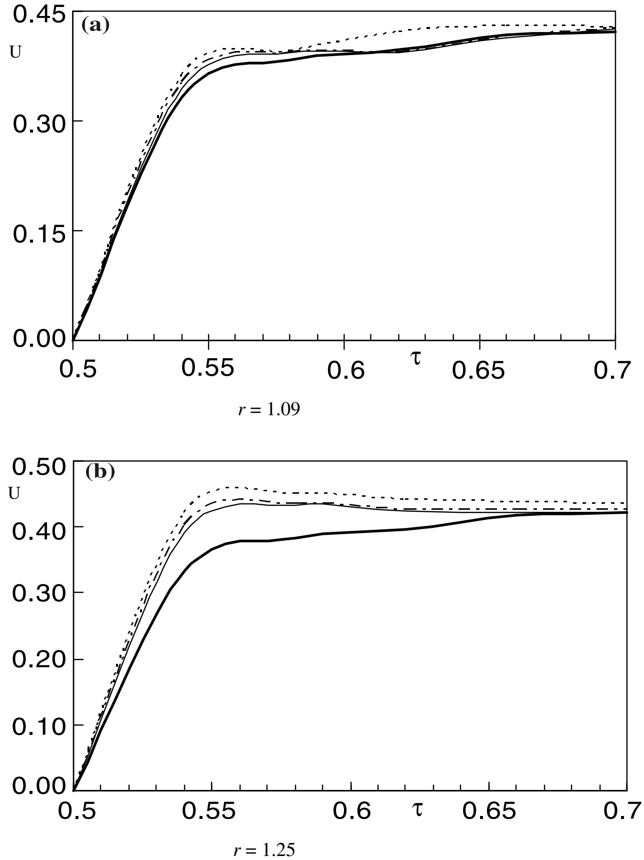

Fig. 4. The neutral stability u–τ curves of different LBE approaches at different grid stretches (uniform flow, $u_{y=0}$; (—-) LBE; (- - - -) DLBE; (——) ISLBM; (— - —) TLLBM.

the LBE, the TLLBM gains better numerical stability at expense of accuracy by, some times severely, introducing hyper-viscosities. Since the LBE approaches are convection-dominated, one can expect that, to be consistent to this nature, the TLLBM with the upwind scheme is more stable than that with central scheme. This can be well observed in Fig. 1. The effects of the grid stretch on the stability of the different LBE approaches were compared in Fig. 4 (a) and (b). In this analysis, we carried our analysis on a non-uniform grid and two space positions with grid stretch ratios of $r = 1.09$ and 1.25, respectively, and only results based on the upwind schemes were presented. As observed from Fig. 4 (a) and (b), the stability behaviors are improved with increase of the grid stretch, and the

DLBE and the TLLBM have demonstrated better stability behaviors than the ISLBM at the same grid stretch.

Figures 3 and 4 indicate that for $\tau > 0.65$, one can have $U > 0.4$. To verify this, we did simulation for a 2D channel flow with $U\,\max = 0.4$ and $\tau = 0.7$, and found that the LBE, ISLBM and TLLBM are all stable in getting convergent solutions.

5. CONCLUSIONS

The stability and Hydrodynamic behaviors of different lattice Boltzmann models were investigated by using the von Neumann linearized analysis in this paper. It is found that the DLBE, the ISLBM and the TLLBM improve the stability behavior of the LBE but deteriorate the hydrodynamic behaviors of the LBE when the non-uniform grid is used. The LBE approaches (except LBE) with the upwind scheme have a better stability and a poorer shear hydrodynamic behavior than those with the central scheme.

We hope that the present work can provide some guidelines for researchers involving different lattice Boltzmann models. Our future work will extend the investigation to fully thermal and compressible LBE approaches in three-dimensional space. Furthermore, finding the ways to improve the hydrodynamic behaviors of the DLBE, the ISLBM and the TLLBM could be a challenging work.

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